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ON QUASI-POLYNOMIAL ALGEBRAS

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

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Introduction

Let R be a commutative ring, and let $R^{[n]}$ be a polynomial ring $R[X_1, \dots, X_n]$. For a prime ideal $\mathfrak{p}$ of R, we denote by $j(\mathfrak{p})$ the integral closure of $R/\mathfrak{p}$ in its quotient field $k(\mathfrak{p}) = R\mathfrak{p}/\mathfrak{p}R_{\mathfrak{p}}$. Note that $j(\mathfrak{p})$ is an R-subalgebra of $k(\mathfrak{p})$. An R-algebra A is called a quasi-polynomial algebra in n-variables if the base extension $j(\mathfrak{p}) \otimes_R A$ is $j(\mathfrak{p})$-isomorphic to a polynomial ring $j(\mathfrak{p})^{[n]}$ in n-variables over $j(\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of R. The purpose of the present paper is to discuss some topics related to quasi-polynomial algebras.

We will need some terminology from [8] and [9]. An R-algebra A is called weakly projective if A is a retract of a polynomial ring $R^{[n]}$, i.e. there is a pair of R-homomorphisms $g: A \to R^{[n]}$ and $f: R^{[n]} \to A$ such that $f \circ g = \mathrm{id}_A$, the identity map on A. An R-algebra A is called invertible[1] if there is an R-algebra B such that $A \otimes_R B \cong_R R^{[n]}$ for some n. An invertible algebra is weakly projective but the converse does not necessarily hold (6.11) (see [14], [22]). An R-algebra A is called strongly projective if there is a projective A-module M such that the symmetric algebra $S_A(M)$ is R-isomorphic to a polynomial ring $R^{[n]}$. These three types of algebras are very closely related to quasi-polynomial algebras.

After Milnor's construction of projective modules, [16], Connell and Wright construct invertible algebras, [9], and Yanik constructs weakly projective algebras, [22]. In a similar way we will construct strongly projective algebras in Section 3, which play important role in this paper.

In Section 2 we consider the automorphism group of $R^{[n]}$ in connection with a lemma (2.5) which is the basic tool in constructing strongly projective algebras.

Theorem 4.5 shows that a finitely-generated, flat, quasi-polynomial algebra A over a noetherian ring R is strongly projective provided that the differential

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[1] Called 'projective' in [8] and [22].

A-module $\Omega_R(A)$ is projective. An R-algebra A is called stably equivalent to an R-algebra B if $A^{[n]} \cong_R B^{[n]}$ for some n, [11]. We obtain the following corollary (4.7) of Theorem 4.5: Let A be a finitely-generated, flat, quasi-polynomial algebra over a noetherian ring R. Then A is stably equivalent to $R^{[n]}$ if and only if $\Omega_R(A)$ is stably equivalent to A^n, i.e. $\Omega_R(A) \oplus A^r \cong A^{n+r}$ for some r. We note that the assumption of $\Omega_R(A)$ to be projective over A can be deleted from Theorem 4.5 when R is reduced (5.2).

In Section 5 we consider a finitely-generated flat R-algebra A such that the fibre ring $k(\mathfrak{p}) \otimes_R A$ is $k(\mathfrak{p})$-isomorphic to $k(\mathfrak{p})^{[1]}$ for each prime ideal $\mathfrak{p}$ of R. When R is noetherian ring, we prove that such an R-algebra A is a locally quasi-polynomial R-algebra, i.e. $A_\mathfrak{p}$ is a quasi-polynomial $R_\mathfrak{p}$-algebra for each prime ideal $\mathfrak{p}$ of R (see [15]).

In Section 6 we consider invertible algebras. If A is an invertible R-algebra, then $\Omega_R(A)$ is finitely-generated projective over A (6.2). The main purpose of this section is to prove the following theorem (6.7): Let R be a commutative ring, and let A be an R-algebra. Then A is an invertible R-algebra such that $\Omega_R(A)$ is of rank one if and only if A is stably equivalent to $S_R(M)$ for some finitely-generated projective R-module M of rank one.

1. Generalities

In this paper all rings are assumed to be commutative with unit.

1.1. Let R be a ring, and let A be an R-algebra with the structure homomorphism $f: R \to A$. We call that A is an augmented R-algebra if there is a ring homomorphism $g: A \to R$ such that $g \circ f = \mathrm{id}_R$. The homomorphism g is called an augmentation of A. In this case, R is said to be a retract of A. If R is a retract of A, then f is injective, and hence R may be viewed as a subring of A. Thus A is an augmented R-algebra if and only if A is of the form $A = R \oplus I$ for some ideal I of A, [10]. An R-algebra A is called weakly projective if A is a retract of a polynomial ring $R^{[n]}$, [8].

1.2. Let M be an A-module. We denote by $S_A^r(M)$ the component of $S_A(M)$ in degree r. The natural augmentation $\phi: S_A(M) \to A$ is defined by $\phi(F) = a$, where F is any element of $S_A(M)$ and a is a constant of F. Therefore $\operatorname{Ker} \phi = \bigoplus_{r>0} S_A^r(M)$, where $\operatorname{Ker} \phi$ denotes the kernel of ϕ.

1.3. An R-algebra A is called strongly projective if there is a projective A-module M such that $S_A(M) \cong_R R^{[n]}$. Note that a strongly projective algebra is weakly projective.

1.4. let N be a finitely-generated projective R-module, and let $A = S_R(N)$. We can choose a projective R-module E so that $N \oplus E = R^n$, a free R-module of rank n. If we set $M = A \otimes_R E$, then M is a projective A-module and we have

$$S_A(M) \cong_A A \otimes_R S_R(E) \cong_R R^{[n]}.$$

This shows that the symmetric algebra A is strongly projective over R.

1.5. Let A be a weakly (resp. strongly) projective R-algebra. Then A is a finitely-generated flat R-algebra. Given an R-algebra S, the S-algebra $S \otimes_R A$ is weakly (resp. strongly) projective. These facts immediately follow from the definition of weakly (resp. strongly) projective algebras. If M is an A-module such that $S_A(M) \cong_R R^{[n]}$, then $S_A(M)$ is finitely-generated over A, and hence M is also finitely-generated as an A-module.

1.6. Given a ring homomorphism $\phi : R \to S$, and given an R-algebra A (resp. R-module N), we often denote by $\phi_\# A$ (resp. $\phi_\# N$) the induced S-algebra $S \otimes_R A$ (resp. S-module $S \otimes_R N$). (See [16,p. 19].) An A-module M is said to be extended from R if there is an R-module M' such that $M \cong A \otimes_R M'$.

1.7. Proposition. *Let A be a weakly projective R-algebra, and let M be an A-module. Suppose the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is extended from $R_\mathfrak{p}$ for each prime ideal* $\mathfrak{p}$ of *R. Then M is extended from T.*

Proof. Since A is a retract of a polynomial ring, there is a pair of R-homomorphisms $h : A \to R^{[n]}$ and $j : R^{[n]} \to A$ such that $j \circ h = \mathrm{id}_A$. Since $M_\mathfrak{p}$ is extended from $R_\mathfrak{p}$, we can choose an $R_\mathfrak{p}$-module N so that $M_\mathfrak{p} \cong A_\mathfrak{p} \otimes_{R_\mathfrak{p}} N$. Therefore we have $R_\mathfrak{p} \otimes_R h_\# M \cong R_\mathfrak{p}^{[n]} \otimes_{R_\mathfrak{p}} N$, i.e. $R_\mathfrak{p} \otimes_R h_\# M$ is an extended $R_\mathfrak{p}^{[n]}$-module from R. This shows that $h_\# M$ is an extended $R^{[n]}$-module from R, [18]. Thus we can find an R-module L so that $h_\# M \cong R^{[n]} \otimes_R L$. This $R^{[n]}$-isomorphism induces an A-isomorphism $j_\# h_\# M \cong j_\#(R^{[n]} \otimes_R L) \cong A \otimes_R L$. On the other hand, $j_\# h_\# M \cong M$ because $j \circ h = \mathrm{id}_A$. Therefore $M \cong A \otimes_R L$, which completes the proof. □

1.8. Proposition. *Let M be an A-module such that $\Omega_R(A) \oplus M$ is a finitely-generated projective A-module of rank n. Then $\Omega_R(S_A(M)) \cong S_A(M) \otimes_A (\Omega_R(A) \oplus M)$.*

Proof. Let $\phi : S_A(M) \to A$ be a natural augmentation, and let $I = \operatorname{Ker} \phi$. Note that $I/I^2 \cong M$ as A-modules because $I = \bigoplus_{r>0} S_A^r(M)$. Now consider the usual exact sequence

$$I/I^2 \xrightarrow{\psi} \phi_\# \Omega_R(S_A(M)) \longrightarrow \Omega_R(A) \longrightarrow (0)$$

of A-modules. Since $\Omega_R(A)$ is projective over A, this exact sequence splits, and therefore $\phi_\# \Omega_R(S_A(M)) = \Omega_R(A) \oplus \psi(I/I^2)$. Let P be a prime ideal of A. Then M_P is a free A_P-module of finite rank, say r. Therefore $S_A(M)_P \cong A_P^{[r]}$ and

$$\Omega_R(S_A(M)_P) \cong S_A(M)_P \otimes_{A_P} (\Omega_R(A_P) \oplus A_P^r).$$

Moreover

$$\Omega_R(A_P) \oplus A_P^r \cong A_P \otimes_A (\Omega_R(A) \oplus M) \cong A_P^n,$$

which implies $\Omega(S_A(M)_P) \cong S_A(M)_P^n$. This shows that $\Omega_R(S_A(M))$ is extended from A (Proposition 1.7) because the symmetric algebra $S_A(M)$ is weakly projective over A (1.4). If we denote by $\phi_P: S_A(M)_P \to A_P$ the induced augmentation, then

$$A_P \otimes_A \phi_* \Omega_R(S_A(M)) \cong \phi_{P*} \Omega_R(S_A(M)_P) \cong A_P^n.$$

So $\phi_* \Omega_R(S_A(M))$ is a projective A-module of rank n. Since $\phi_* \Omega_R(S_A(M))$ is a homomorphic image of $\Omega_R(A) \oplus M$, we get $\phi_* \Omega_R(S_A(M)) \cong \Omega_R(A) \oplus M$. Thus

$$\Omega_R(S_A(M)) \cong S_A(M) \otimes_A \phi_* \Omega_R(S_A(M)) \cong S_A(M) \otimes_A (\Omega_R(A) \oplus M)$$

and we conclude the proof. □

1.9. Corollary. *Let M be a projective A-module such that $S_A(M) \cong_R R^{[n]}$. Then $\Omega_R(A) \oplus M \cong A^n$.*

1.10. Proposition. *Let A be an R-algebra such that $S_A(M)$ is a strongly projective R-algebra for some finitely-generated projective A-module M. Then A is also a strongly projective R-algebra.*

Proof. Let us set $B = S_A(M)$. We can choose a projective B-module L so that $S_B(L) \cong_R R^{[n]}$. It follows from Corollary 1.9 that $\Omega_R(B) \oplus L \cong B^n$. Thus $\Omega_R(B)$ is a finitely-generated projective B-module. Given a prime ideal P of A, as shown in the proof of Proposition 1.8, $\Omega_R(B)_P$ is a free B_P-module of finite rank, say r. If we set $N = B^n \oplus L$, then

$$N_P = B_P^{n-r} \oplus \Omega_R(B)_P \oplus L_P \cong B_P^{2n-r},$$

i.e. N_P is an extended $S_A(M)_P$-module from A_P. Therefore N is extended from A by Proposition 1.7, and hence we can choose some projective A-module E so that $N \cong B \otimes_A E$. As a consequence we obtain $S_B(N) \cong_B B \otimes_A S_A(E) \cong_A S_A(M \oplus E)$. On the other hand, $S_B(N) \cong_B S_B(B^n \oplus L) \cong_R R^{[2n]}$. So $S_A(M \oplus E) \cong_R R^{[2n]}$, which shows that A is strongly projective over R. □

1.11. Proposition. *Let A be a strongly projective R-algebra. Then there is an injection $\theta: A \hookrightarrow R^{[n]}$ such that $A^{[n]} \cong_R S_{R^{[n]}}(\theta_* \Omega_R(A))$.*

Proof. Let M be a projective A-module such that $S_A(M) \cong_R R^{[n]}$. under this isomorphism we identify $S_A(M)$ with $R^{[n]}$. In particular A is an R-subalgebra of $R^{[n]}$. Since $S_A(M) \otimes_A S_A(\Omega_R(A)) \cong_A A^{[n]}$ (Corollary 1.9), we have

$$A^{[n]} \cong_R R^{[n]} \otimes_A S_A(\Omega_R(A)) \cong_R S_{R^{[n]}}(R^{[n]} \otimes_A \Omega_R(A)),$$

which completes the proof. □

1.12. A ring R is said to be a PE_m-ring if an arbitrary projective $R^{[n]}$-module M of rank m is extended from R for any positive integer n (cf. [12]). In particular a

PE_1-ring is also said to be a seminormal ring. Namely a ring R is seminormal if and only if a natural injection $\mathrm{Pic}(R) \to \mathrm{Pic}(R^{[n]})$ is an isomorphism for any positive integer n, where $\mathrm{Pic}(*)$ denotes the Picard group (see [20], [21]). A one-dimensional noetherian domain D is a PE_m-ring for any positive integer m if and only if D is seminormal ([6]).

1.13. Proposition. *Let R be a PE_m-ring, and let A be a strongly projective R-algebra. Suppose $\Omega_R(A)$ is projective of rank m. Then A is stably equivalent to $S_R(N)$ for some projective R-module N.*

Proof. By virtue of Proposition 1.11, A may be considered as an R-subalgebra of $R^{[n]}$ such that $A^{[n]} \cong_R S_{R^{[n]}}(R^{[n]} \otimes_A \Omega_R(A))$. Since $\Omega_R(A)$ is projective over A of rank m, $R^{[n]} \otimes_A \Omega_R(A)$ is extended from R, and hence we can choose some projective R-module N so that $R^{[n]} \otimes_A \Omega_R(A) \cong R^{[n]} \otimes_R N$. This shows that

$$A^{[n]} \cong_R S_{R^{[n]}}(R^{[n]} \otimes_R N) \cong S_R(N)^{[n]}$$

as required. $\square$

2. Automorphisms of polynomial rings

2.1. Let R be a ring. After [9] we will write $\mathrm{GA}_n(R)$ for the group of R-automorphisms of $R^{[n]}$. Let $(F_i)_i^n = (F_1, \ldots, F_n)$ denote the vector for $F_1, \ldots, F_n \in R^{[n]}$. An element $\alpha \in \mathrm{GA}_n(R)$ is called elementary if α is of the form

$$(\alpha(X_i))_i^n = (X_1, \ldots, X_{j-1}, X_j + F, X_{j+1}, \ldots, X_n),$$

where $F \in R[X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n]$. The subgroup of $\mathrm{GA}_n(R)$ generated by all elementary elements is denoted by $\mathrm{EA}_n(R)$.

2.2. Let $\mathrm{GL}_n(R)$ be a general linear group, and let I_n be an identity matrix of $\mathrm{GL}_n(R)$. We denote by $E_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices.

2.3. Let α be an element of $\mathrm{GA}_n(R)$. The Jacobian matrix $J(\alpha) \in \mathrm{GL}_n(R^{[n]})$ is defined by $J(\alpha) = (\partial\alpha(X_j)/\partial X_i)_{ij}$. Let $\phi = (f_{ij})$ be an element of $\mathrm{GL}_m(R^{[n]})$, and let r be an integer such that $r \geq n$. We define $\pi_r(\phi) \in \mathrm{GA}_{m+r}(R)$ by

$$(\pi_r(\phi)X_i))_i^{r+m} = (X_1, \ldots, X_{r+m}) \begin{pmatrix} I_r & 0 \\ 0 & \phi \end{pmatrix},$$

i.e. $\pi_r(\phi)(X_i) = X_i$ $(i = 1, \ldots, r)$ and

$$\pi_r(\phi)(X_{r+j}) = \sum_{i=1}^{m} f_{ij} X_{r+i} \quad (j = 1, \ldots, m).$$

Note that $\pi_r: \mathrm{GL}_m(R^{[n]}) \to \mathrm{GA}_{m+r}(R)$ is an injective group homomorphism. Now let α' be an element of $\mathrm{GA}_{n+1}(R)$ defined by $(\alpha'(X_i))_i^{n+1} = (\alpha(X_1), \dots, \alpha(X_n), X_{n+1})$. Then the map $\mathrm{GA}_n(R) \to \mathrm{GA}_{n+1}(R)$ defined by $\alpha \to \alpha'$ is clearly an injective group homomorphism. Thus we may regard $\mathrm{GA}_n(R)$ as a subgroup of $\mathrm{GA}_{n+1}(R)$. In particular, $\mathrm{GA}_n(R)$ is a subgroup of $\mathrm{GA}_{2n}(R)$. Therefore $\tilde{\alpha} = \pi_n(J(\alpha)^{-1})\alpha$ is well defined as an element of $\mathrm{GA}_{2n}(R)$. Indeed, $\tilde{\alpha}$ can be described as follows:

$$(\tilde{\alpha}(X_i))_i^{2n} = (\alpha(X_1), \dots, \alpha(X_n), X_{n-1}, \dots, X_{2n}) \begin{pmatrix} I_n & 0 \\ 0 & J(\alpha)^{-1} \end{pmatrix}.$$

(See [9].)

2.4. Given a ring homomorphism $f: R \to \bar{R}$, let us denote by $f^*: \mathrm{GA}_n(R) \to \mathrm{GA}_n(\bar{R})$ (resp. $f^\circ: \mathrm{GL}_n(R) \to \mathrm{GL}_n(\bar{R})$), the induced group homomorphism. If f is injective, f^* (resp. f°) is also injective, and hence $\mathrm{GA}_n(R)$ (resp. $\mathrm{GL}_n(R)$) may be considered as a subgroup of $\mathrm{GA}_n(\bar{R})$ (resp. $\mathrm{GL}_n(\bar{R})$) when R is a subring of $\bar{R}$. If f is surjective, then it is easy to see that $f^*(\mathrm{EA}_n(R)) = \mathrm{EA}_n(\bar{R})$ (resp. $f^\circ(E_n(R)) = E_n(\bar{R})$). In other words, any element α of $\mathrm{EA}_n(\bar{R})$ (resp. $E_n(\bar{R})$) can be lifted to an element of $EA_n(R)$ (resp. $E_n(R)$).

The following, due to [1], [9] and [22], is the key lemma in this paper.

2.5. Lemma. *Let $\bar{R} = R/sR$ where $s \in R$, and let $f: R \to \bar{R}$ be the canonical projection. Suppose $\alpha \in \mathrm{GA}_n(\bar{R})$. Then there is an element $\varrho \in \mathrm{GA}_{2n}(R)$ such that $\tilde{\alpha} = f^*(\varrho)$. If s is a non-zero divisor, then we can choose such a ϱ to be an element of $\mathrm{GA}_{2n}(R) \cap \mathrm{EA}_{2n}(R[\frac{1}{s}])$.*

Proof. See [22, p. 345]. See also [9, p. 156] and [1, p. 466]. □

2.6. Let $f_\lambda: R_\lambda \to \bar{R}$ $(\lambda = 1, 2)$ be a pair of ring homomorphisms, and let $\bar{R}_\lambda = f_\lambda(R_\lambda)$. We say that $\{f_1, f_2\}$ is a Γ-pair if $\{f_1, f_2\}$ satisfies the following conditions:

(Γ_1) R_2 is noetherian.

(Γ_2) $\operatorname{Ker} f_2$ is a principal ideal.

(Γ_3) There is a unit element t of $\bar{R}$ such that $t \in \bar{R}_1 \cap \bar{R}_2$ and $\bar{R} = \bar{R}_2[\frac{1}{t}] = \bar{R}_1 + t\bar{R}_2$.

From the condition (Γ_3) it easily follows that $\bar{R} = \bar{R}_1 + t^r \bar{R}_2$ for any positive integer r. The following proposition is due essentially to Bryński [7].

2.7. Proposition. *Let $\{f_1, f_2\}$ be a Γ-pair as in* 2.6. *Then:*

(i) $\mathrm{EA}_n(\bar{R}) \subset \mathrm{GA}_n(\bar{R}_2)\mathrm{EA}_n(\bar{R}_1)$,

(ii) $E_n(\bar{R}) \subset \mathrm{GL}_n(\bar{R}_2)E_n(\bar{R}_1)$.

Proof. The proof of (ii) is quite similar to the proof of (i) and we only prove (i). Let β be an elementary element of $\mathrm{GA}_n(\bar{R})$ which is of the form

$$(\beta(X_i))_i^n = (X_1, \dots, X_{j-1}, X_j + F, X_{j+1}, \dots, X_n)$$

for $F \in \bar{R}[X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n]$. Let r be a positive integer. Since $\bar{R} = \bar{R}_1 + t^r \bar{R}_2$, we can choose a pair of elements $F_\lambda \in \bar{R}_\lambda[X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n]$ $(\lambda = 1, 2)$ so that $F = F_1 + t^r F_2$. We define an elementary element $\beta_{(r)}$ of $\mathrm{GA}_n(\bar{R}_1)$ by

$$(\beta_{(r)}(X_i))_i^n = (X_1, \dots, X_{j-1}, X_j + F_1, X_{j+1}, \dots, X_n).$$

Now let α be any element of $\mathrm{EA}_n(\bar{R})$. Then α is of the form $\alpha = \alpha_m \cdots \alpha_1$, where $\alpha_1, \dots, \alpha_m$ are elementary elements. Let us set $\alpha_{kr} = \alpha_k \cdots \alpha_1 \alpha_{1(r)}^{-1} \cdots \alpha_{k(r)}^{-1}$. If $\deg H$ denotes the total degree of $H \in \bar{R}^{[n]}$, then there is an integer u such that $\deg \alpha_{kr}(X_i) \le u$ for $k = 1, \dots, m$, $i = 1, \dots, n$, $r = 1, 2, 3, \dots$. Given a positive integer v, we shall show $\alpha_{kr}(X_i) \equiv X_i \pmod{t^v \bar{R}_2^{[n]}}$ for some large integer r. Suppose $\alpha_{k-1,r}(X_i)$ is of the form $\alpha_{k-1,r}(X_i) = X_i + t^v H_i$ for $H_i \in \bar{R}_2^{[n]}$ $(i = 1, \dots, n)$. Since $\bar{R} = \bar{R}_2[\frac{1}{t}]$ and $\deg H_i \le u$, given an integer w, we can easily verify that $\alpha_{kr}(X_i) \equiv X_i \pmod{t^w \bar{R}_2^{[n]}}$ for sufficiently large integers r and v. This shows that $\alpha_{kr}(X_i) \equiv X_i \pmod{t^v \bar{R}_2^{[n]}}$ for $k = 1, \dots, n$ by induction. In particular $\alpha_{mr} \in \mathrm{GA}_n(\bar{R}_2^{[n]})$, and hence $\alpha \in \mathrm{GA}_n(\bar{R}_2)\mathrm{EA}_n(\bar{R}_1)$, which completes the proof of (*i*). $\square$

2.8. Proposition. *Let $\{f_1, f_2\}$ be a Γ-pair as in* 2.6, *and let ϕ be any element of* $\mathrm{GL}_n(\bar{R})$. *Then there is an element $\psi \in \mathrm{GL}_{3n}(\bar{R})$ such that $\phi \oplus \psi \in E_{4n}(\bar{R}_2)E_{4n}(\bar{R}_1)$.*

Proof. According to [16, p. 22], we have $\phi \oplus \phi^{-1} \in E_{2n}(\bar{R})$. Since

$$E_{2n}(\bar{R}) \subset \mathrm{GL}_{2n}(\bar{R}_2)E_{2n}(\bar{R}_1)$$

by Proposition 2.7(ii), we can choose $\phi_1 \in E_{2n}(\bar{R}_1)$ and $\phi_2 \in \mathrm{GL}_{2n}(\bar{R}_2)$ so that $\phi \oplus \phi^{-1} = \phi_2 \phi_1$. Again we have $\phi_2 \oplus \phi_2^{-1} \in E_{4n}(\bar{R}_2)$, and hence

$$\phi \oplus \phi^{-1} \oplus \phi_2^{-1} = \phi_2 \phi_1 \oplus \phi_2^{-1} = (\phi_2 \oplus \phi_2^{-1})(\phi_1 \oplus I_{2n}) \in E_{4n}(\bar{R}_2)E_{4n}(\bar{R}_1).$$

If we set $\psi = \phi^{-1} \oplus \phi_2^{-1}$, then $\psi \in \mathrm{GL}_{3n}(\bar{R})$ and $\phi \oplus \psi \in E_{4n}(\bar{R}_2)E_{4n}(\bar{R}_1)$ as required. $\square$

2.9. Proposition. *Let $\{f_1, f_2\}$ be a Γ-pair as in* 2.6.

(i) *For any $\alpha \in \mathrm{GA}_n(\bar{R}_2)$, we have $\tilde{\alpha} \in f_2^*(\mathrm{GA}_{2n}(R_2))$.*

(ii) *For any $\alpha \in \mathrm{GA}_n(\bar{R})$, we have $\tilde{\alpha} \in \mathrm{GA}_{2n}(\bar{R}_2)\mathrm{EA}_{2n}(\bar{R}_1)$.*

Proof. (i): This is an immediate consequence of Lemma 2.5.

(ii): Let T be an indeterminate, and let $g : \bar{R}[T] \to \bar{R}$ be an augmentation defined by $g(T) = 0$. There is an element $\varrho \in \mathrm{GA}_{2n}(\bar{R}[T]) \cap \mathrm{EA}_{2n}(\bar{R}[T, T^{-1}])$ such that $\tilde{\alpha} = g^*(\varrho)$ (Lemma 2.5). This shows that $\tilde{\alpha}\varrho^{-1}$ is of the form $(\tilde{\alpha}\varrho^{-1}(X_i))_i^{2n} = (X_i + TH_i)_i^{2n}$, where $H_i = H_i(T, X_1, \dots, X_{2n}) \in \bar{R}[T]^{[2n]}$. Recall that $\bar{R} = \bar{R}_2[\frac{1}{t}]$, and therefore we get $t^r H_i(t^r, X_1, \dots, X_{2n}) \in \bar{R}_2^{[2n]}$ for sufficiently large integer r. Let $h_r : \bar{R}[T, T^{-1}] \to \bar{R}$ is an augmentation defined by $h_r(T) = t^r$. Since $h_r^*(\tilde{\alpha}\varrho^{-1}) = \tilde{\alpha}h_r^*(\varrho^{-1})$, we have

$$(\tilde{\alpha}h_r^*(\varrho^{-1})(X_i))_i^{2n} = (X_i + t^r H_i(t^r, X_1, \dots, X_{2n}))_i^{2n},$$

which implies $\tilde{\alpha}h_r^*(\varrho^{-1}) \in \mathrm{GA}_{2n}(\bar{R}_2)$. On the other hand, from the fact $\varrho \in$

$EA_{2n}(\bar{R}[T, T^{-1}])$ it follows that $h_r^*(\varrho) \in EA_{2n}(\bar{R})$, and hence $h_r^*(\varrho) \in GA_{2n}(\bar{R}_2)EA_{2n}(\bar{R}_1)$ by Proposition 2.7. Thus $\tilde{\alpha} \in GA_{2n}(\bar{R}_2)EA_{2n}(\bar{R}_1)$, which completes the proof of (ii). $\square$

3. The fibre product of strongly projective algebras

As for the basic concepts and results on fibre products we refer to [16, §2] and [3, Ch. VIII, §3].

3.1. Consider a pullback diagram

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & R_2 \arrow[d, "f_2"] \\ R_1 \arrow[r, "f_1"] & \bar{R} \end{tikzcd} \tag{1}$$

of rings. Let A_λ be a strongly projective R_λ-algebra for $\lambda = 1, 2$. Suppose there is an $\bar{R}$-isomorphism $\alpha : f_{1\#}A_1 \to f_{2\#}A_2$. The fibre product $A = AL_R(A_1, \alpha, A_2)$ is defined by the diagram

$$\begin{tikzcd} & A_2 \arrow[d, "f_2'"] \\ A_1 \arrow[r, "\alpha \circ f_1'"] & f_{2\#}A_2 \end{tikzcd} \tag{2}$$

where $f_\lambda' : A_\lambda \to f_{\lambda\#}A_\lambda$ denotes the natural map. We make A into an R-algebra by $b(a_1, a_2) = (p_1(b)a_1, p_2(b)a_2)$ for $b \in R$ and $(a_1, a_2) \in A$. (See [16, p. 20] and [22].) For convenience sake, we put $F_1 = \alpha \circ f_1'$, $F_2 = f_2'$ and $\bar{A} = f_{2\#}A_2$. Then the diagram (2) induces the following pullback diagram of R-algebras:

$$\begin{tikzcd} A \arrow[r, "P_2"] \arrow[d, "P_1"'] & A_2 \arrow[d, "F_2"] \\ A_1 \arrow[r, "F_1"] & \bar{A} \end{tikzcd} \tag{3}$$

where P_λ denotes the projection. Let M_λ be a finitely-generated projective A_λ-module. Suppose there is an $\bar{A}$-isomorphism $\Phi : F_{1\#}M_1 \to F_{2\#}M_2$. The fibre product

$M = M_A(M_1, \Phi, M_2)$ is defined by the diagram

$$\begin{array}{ccc} & & M_2 \\ & & \Big\downarrow{\scriptstyle F_2'} \\ M_1 & \xrightarrow{\Phi \circ F_1'} & F_{2\#}M_2 \end{array} \tag{4}$$

where $F_\lambda' : M_\lambda \to F_{\lambda\#}M_\lambda$ denotes the natural map. Note that M is an A-module, [16, p. 20]. According to [5, Ch. III, §5], the diagram (4) induces a diagram

$$\begin{array}{ccc} & & S_{A_2}(M_2) \\ & & \Big\downarrow{\scriptstyle S(F_2')} \\ S_{A_1}(M_1) & \xrightarrow{S(\Phi \circ F_1')} & S_{\bar{A}}(\bar{M}) \end{array} \tag{5}$$

of symmetric algebras, where $\bar{M} = F_{2\#}M_2$. Let $M^S = M_A(M_1, \Phi, M_2)^S$ denote the fibre product defined by the diagram (5). Given a pair of elements $a = (a_1, a_2) \in A$ and $c = (c_1, c_2) \in M^S$, we can easily verify that $(a_1c_1, a_2c_2) \in M^S$, and hence M^S may be viewed as an A-algebra by setting $ac = (a_1c_1, a_2c_2)$. Now let

$$\bar{\Phi} : F_{1\#}S_{A_1}(M_1) \to F_{2\#}S_{A_2}(M_2)$$

be an induced $\bar{A}$-isomorphism from $\Phi : F_{1\#}M_1 \to F_{2\#}M_2$. Then the fibre product $\mathrm{AL}_A(S_{A_1}(M_1), \bar{\Phi}, S_{A_2}(M_2))$ is well defined. If we identify $F_{\lambda\#}S_{A_\lambda}(M_\lambda)$ with $S_{\bar{A}}(F_{\lambda\#}M_\lambda)$ by a natural isomorphism $F_{\lambda\#}S_{A_\lambda}(M_\lambda) \cong S_{\bar{A}}(F_{\lambda\#}M_\lambda)$, then $\bar{\Phi} = S(\Phi)$. Therefore we have $M^S = \mathrm{AL}_A(S_{A_1}(M_1), \bar{\Phi}, S_{A_2}(M_2))$. We keep these notations fixed throughout this section.

3.2. We consider the special case where $A_\lambda = R_\lambda^{[n]}$. We identify $f_{\lambda\#}A_\lambda$ with $\bar{R}^{[n]}$ by an obvious isomorphism. In this case, $f_\lambda' : R_\lambda^{[n]} \to R^{[n]}$ is a natural homomorphism defined by $f_\lambda'(X_i) = X_i$ $(i = 1, \ldots, n)$ and α is an element of $\mathrm{GA}_n(\bar{R})$. Suppose $\alpha \in f_2^*(\mathrm{GA}_n(R_2)) f_1^*(\mathrm{GA}_n(R_1))$. We can choose a pair of elements $\alpha_\lambda \in \mathrm{GA}_n(R_\lambda)$ $(\lambda = 1, 2)$ so that $\alpha = f_2^*(\alpha_2) f_1^*(\alpha_1^{-1})$. If we set $Y_i = \alpha_1(X_i)$, $Z_i = \alpha_2(X_i)$ and $V_i = f_2^*(\alpha_2)(X_i)$ $(i = 1, \ldots, n)$, then $R_1^{[n]} = R_1[Y_1, \ldots, Y_n]$, $R_2^{[n]} = R_2[Z_1, \ldots, Z_n]$ and $\bar{R}^{[n]} = \bar{R}[V_1, \ldots, V_n]$. Furthermore $A = \mathrm{AL}_n(R_1^{[n]}, \alpha, R_2^{[n]})$ is the fibre product defined by the diagram

$$\begin{array}{ccc} & & R_2[Z_1, \ldots, Z_n] \\ & & \Big\downarrow{\scriptstyle F_1} \\ R_1[Y_1, \ldots, Y_n] & \xrightarrow{F_2} & \bar{R}[V_1, \ldots, V_n] \end{array} \tag{6}$$

where $F_1(Y_i) = F_2(Z_i) = V_i$ $(i = 1, \ldots, n)$. Note that $A \cong_R R^{[n]}$.

3.3. In a similar way of 3.2 we consider the case where $M_\lambda = A_\lambda^n$ for any $A = \mathrm{AL}_R(A_1, \alpha, A_2)$. Give a ring B, let $\{e_1, \dots, e_n\}$ be a basis of B^n. Thus $A_\lambda^n = A_\lambda e_1 \oplus \cdots \oplus A_\lambda e_n$ and $\bar{A} = \bar{A}e_1 \oplus \cdots \oplus \bar{A}e_n$. We identify $F_{\lambda\#}A^n$ with $\bar{A}^n$ by an obvious isomorphism. Note that Φ is an element of $\mathrm{GL}_n(\bar{A})$. Now suppose

$$\Phi \in F_2^\circ(\mathrm{GL}_n(A_2))F_1^\circ(\mathrm{GL}_n(A_1)).$$

Then Φ is of the form $\Phi = F_2^\circ(\Phi_2)F_1^\circ(\Phi_1^{-1})$ for $\Phi_\lambda \in \mathrm{GL}_n(A_\lambda)$. Let us set $\Phi_1(e_i) = y_i$, $\Phi_2(e_i) = z_i$ and $F_2^\circ(\Phi_2)(e_i) = v_i$. The fibre product $M = M_A(A_1^n, \Phi, A_2^n)$ is defined by the following pullback diagram

$$\begin{tikzcd}
M \arrow[r, "G_2"] \arrow[d, "G_1"'] & A_2z_1 \oplus \cdots \oplus A_2z_n \arrow[d, "F_2'"] \\
A_1y_1 \oplus \cdots \oplus A_ny_n \arrow[r, "\Phi \circ F_1'"] & \bar{A}v_1 \oplus \cdots \oplus \bar{A}v_n
\end{tikzcd} \tag{7}$$

where $\Phi \circ F_1'(y_i) = F_2'(z_i) = v_i$. Therefore we may assume $M = A^n$, $G_1(e_i) = y_i$ and $G_2(e_i) = z_i$.

3.4. Proposition. *There is $M = M_A(M_1, \Phi, M_2)$ such that $S_{A_\lambda}(M_\lambda) \cong_{R_\lambda} R_\lambda^{[n]}$.*

Proof. From the definition of F_λ, we get a natural isomorphism $\Omega_R(\bar{A}) \cong F_{\lambda\#}\Omega_{R_\lambda}(A_\lambda)$; so $F_{1\#}\Omega_{R_1}(A_1) \cong F_{2\#}\Omega_{R_2}(A_2)$. Since A_1 and A_2 are strongly projective, we can take finitely-generated modules Q_1 and Q_2 over A_1 and A_2 respectively so that $S_{A_1}(Q_1) \cong_{R_1} R_1^{[m]}$ and $S_{A_2}(Q_2) \cong_{R_2} R_2^{[k]}$. From Corollary 1.9 it follows that $\Omega_{R_1}(A_1) \oplus Q_1 \cong A_1^m$ and $\Omega_{R_2}(A_2) \oplus Q_2 \cong A_2^k$. Therefore if we set $M_1 = Q_1 \oplus A_1^k$ and $M_2 = Q_2 \oplus A_2^m$, then we obtain an isomorphism $\Phi : F_{1\#}M_1 \cong F_{2\#}M_2$ (see [16, p. 22]), i.e. $M = M_A(M_1, \Phi, M_2)$ is well defined. Defining $n = k + m$, it follows that $S_{A_\lambda}(M_\lambda) \cong_{R_\lambda} R_\lambda^{[n]}$ as required. □

3.5. Proposition. *Suppose $\{f_1, f_2\}$ is a Γ-pair. Then $\{F_1, F_2\}$ is a Γ-pair.*

Proof. The proof consists of two steps.

(I) First we shall consider the special case where $A_\lambda = R_\lambda^{[n]}$. As shown in 3.2 we may assume $\bar{A} = \bar{R}^{[n]}$ and $\alpha \in \mathrm{GA}_n(\bar{R})$. In order to prove that $\{F_1, F_2\}$ is a Γ-pair, it suffices to show that $\bar{R}^{[n]} = \alpha(\bar{R}_1^{[n]}) + t\bar{R}_2^{[n]}$. Let $X_1^{v_1} \cdots X_n^{v_n} = H$ be any monomial of $\bar{R}^{[n]}$. We can choose an element $L \in \bar{R}^{[n]}$ so that $\alpha^{-1}(L) = H$. Since $\{f_1, f_2\}$ is a Γ-pair, we have $\bar{R} = \bar{R}_2[\frac{1}{t}]$, and hence $t^rL \in t\bar{R}_2^{[n]}$ for sufficiently large integer r. This shows that $t^rH \in t\bar{R}_2[\alpha^{-1}(X_1), \dots, \alpha^{-1}(X_n)]$. On the other hand, each element $a \in \bar{R}$ is of the form $a = a_1 + t^ra_2$ $(a_\lambda \in \bar{R}_\lambda)$. Therefore aH is contained in $\bar{R}_1^{[n]} + t\bar{R}_2[\alpha^{-1}(X_1), \dots, \alpha^{-1}(X_n)]$, which implies that $\bar{R}^{[n]} = \bar{R}_1^{[n]} + t\bar{R}_2[\alpha^{-1}(X_1), \dots, \alpha^{-1}(X_n)]$. Since $\bar{R}^{[n]} = \alpha(\bar{R}^{[n]})$, we have $\bar{R}^{[n]} = \alpha(\bar{R}_1^{[n]}) + t\bar{R}_2^{[n]}$ as required.

(II) Now we consider the general case. Let $M = M_A(M_1, \Phi, M_2)$ be as in Proposition 3.4. Since $f_\lambda' : A_\lambda \to f_{\lambda\#}A_\lambda$ is a natural map, we obtain a natural isomorphism

$f'_{\lambda\#}S_{A_\lambda}(M_\lambda) \cong f_{\lambda\#}S_{A_\lambda}(M_\lambda)$; in other words,

$$(\bar{R} \otimes_{R_\lambda} A_\lambda) \otimes_{A_\lambda} S_{A_\lambda}(M_\lambda) \cong \bar{R} \otimes_{R_\lambda} S_{A_\lambda}(M_\lambda).$$

Thus

$$S_{\bar{A}}(F_{1\#}M_1) \cong F_{1\#}S_{A_1}(M_1) \cong \alpha_\# f_{1\#}S_{A_1}(M_1)$$

and

$$S_{\bar{A}}(F_{2\#}M_2) \cong F_{2\#}S_{A_2}(M_2) \cong f_{2\#}S_{A_2}(M_2).$$

Under these isomorphisms we assume $S_{\bar{A}}(F_{1\#}M_1) = \alpha_\# f_{1\#}S_{A_1}(M_1)$ and $S_{\bar{A}}(F_{2\#}M_2) = f_{2\#}S_{A_2}(M_2)$. Let $\bar{\alpha}: f_{1\#}S_{A_1}(M_1) \to \alpha_\# f_{1\#}S_{A_1}(M_1)$ be a natural isomorphism induced from $\alpha: f_{1\#}A_1 \to f_{2\#}A_2$. On the other hand the isomorphism $\Phi: F_{1\#}M_1 \to F_{2\#}M_2$ induces an $\bar{A}$-algebra isomorphism $S(\Phi): S_{\bar{A}}(F_{1\#}M_1) \to S_{\bar{A}}(F_{2\#}M_2)$. Therefore if we denote by $\bar{f}_\lambda: S_{A_\lambda}(M_\lambda) \to f_{\lambda\#}S_{A_\lambda}(M_\lambda)$ the natural map, then $S(\Phi \circ F'_1) = S(\Phi) \circ \bar{\alpha} \circ \bar{f}_1$ and $S(F'_2) = \bar{f}_2$. Recall that $S_{A_\lambda}(M_\lambda) \cong_{R_\lambda} R_\lambda^{[n]}$ and $f_{\lambda\#}S_{A_\lambda}(M_\lambda) \cong f_{\lambda\#}R_\lambda^{[n]}$. Thus we may assume $S_{A_\lambda}(M_\lambda) = R_\lambda^{[n]}$ and $f_{\lambda\#}S_{A_\lambda}(M_\lambda) = \bar{R}^{[n]}$. In this case, $f_\lambda: R_\lambda^{[n]} \to \bar{R}^{[n]}$ is a natural homomorphism defined by $\bar{f}_\lambda(X_i) = X_i$. Moreover $S(\Phi) \circ \bar{\alpha}$ is an $\bar{R}$-automorphism of $\bar{R}^{[n]}$, i.e. $S(\Phi) \circ \bar{\alpha}$ is an element of $\mathrm{GA}_n(\bar{R})$. From the assertion (I) it follows that $\{S(\Phi \circ F'_1), S(F'_2)\} = \{S(\Phi) \circ \bar{\alpha} \circ \bar{f}_1, \bar{f}_2\}$ is a Γ-pair. So we get

$$S_{\bar{A}}(\bar{M}) = S(\Phi \circ F'_1)(S_{A_1}(M_1)) + tS(F'_2)(S_{A_2}(M_2)),$$

which implies $\bar{A} = F_1(A_1) + tF_2(A_2)$ because the restriction of $S(\Phi \circ F'_1)$ and $S(F'_2)$ to A_1 and A_2 are F_1 and F_2 respectively. Thus the proof of the proposition is completed. □

3.6. Proposition. *If $\{f_1, f_2\}$ is a Γ-pair, then M is a finitely-generated projective A-module.* (See [16, p. 20].)

Proof. According to [16, p. 22], we can choose $N = M_A(N_1, \bar{\Phi}, N_2)$ so that $M_\lambda \oplus N_\lambda \cong A_\lambda^n$. Since $M \oplus N = M_A(M_1 \oplus N_1, \Phi \oplus \bar{\Phi}, M_2 \oplus N_2)$, we may assume $M_\lambda = A_\lambda^n$ and $\Phi \in \mathrm{GL}_n(\bar{A})$ (3.3). Note that $\{F_1, F_2\}$ is a Γ-pair by Proposition 3.5, and we can choose $\Psi \in \mathrm{GL}_{3n}(\bar{A})$ so that $\Phi \oplus \Psi \in E_{4n}(F_2(A_2))E_{4n}(F_1(A_1))$ by Proposition 2.8. If we set $L = M_A(A_1^{3n}, \Psi, A_2^{3n})$, then $M \oplus L \cong M_A(A_1^{4n}, \Phi \oplus \Psi, A_2^{4n})$. Since $E_{4n}(F_\lambda(A_\lambda)) = F_\lambda^\circ(E_{4n}(A_\lambda))$, we have $M \oplus L = A^{4n}$ (3.3), which proves the proposition. □

3.7. Proposition. (Milnor, [16, p. 20].) *Every finitely-generated projective module over A is isomorphic to $M = M_A(M_1, \Phi, M_2)$ for some M_1, M_2 and Φ.*

Proof. Let L be a finitely-generated projective A-module. We can choose some A-module K so that $L \oplus K \cong A^n$ for some n. Let us set $M_\lambda = P_{\lambda\#}L$ and $N_\lambda = P_{\lambda\#}K$. Since $F_1 \circ P_1 = F_2 \circ P_2$, there are natural isomorphisms $\Phi: F_{1\#}M_1 \to F_{2\#}M_2$ and $\Psi: F_{1\#}N_1 \to F_{2\#}N_2$. If we define $M = M_A(M_1, \Phi, M_2)$ and $N = M_A(N_1, \Psi, N_2)$, then $M \oplus N = M_A(M_1 \oplus N_1, \Phi \oplus \Psi, M_2 \oplus N_2)$. Now the natural map $L \oplus K \to M \oplus N$ is clearly an isomorphism, and hence the natural map $L \to M$ is also an isomorphism. □

3.8. Proposition. *Suppose $\{f_1, f_2\}$ is a Γ-pair. Then M_λ is naturally isomorphic to $P_{\lambda\#}M$, i.e. the A_λ-homorphism $\phi_\lambda : P_{\lambda\#}M \to M_\lambda$ induced from the projection $\bar{P}_\lambda : M \to M_\lambda$ is an isomorphism.*

Proof. The proof is similar to the proof of Theorem 2.3 in [16, p. 20]. The proof consists of two steps.

(I) Suppose $M_\lambda = A_\lambda^n$ and $\Phi \in F_2^\circ(\mathrm{GL}_n(A_2))F_1^\circ(\mathrm{GL}_n(A_1))$. Then M is defined by the pullback diagram (7) (3.3). Therefore $P_{1\#}M = A_1e_1 \oplus \cdots \oplus A_1e_n$ and $M_1 = A_1y_1 \oplus \cdots \oplus A_1y_n$. Furthermore $\bar{P}_1 : M \to M_1$ is defined by $\bar{P}_1(e_i) = G_1(e_i) = y_i$. This shows that $\phi_1 : P_{1\#}M \to M_1$ is defined by $\phi_1(e_i) = y_i$, and hence ϕ_1 is an isomorphism.

(II) We consider the general case. From the proof of Proposition 3.6, we may regard $M = M_A(M_1, \Phi, M_2)$ as a direct summand of $M_A(M_1 \oplus L_1, \Phi \oplus \Psi, M_2 \oplus L_2)$ for some $L = M_A(L_1, \Psi, L_2)$ which satisfies $M_\lambda \oplus L_\lambda = A_\lambda^n$ and

$$\Phi \oplus \Psi \in F_2^\circ(E_n(A_2))F_1^\circ(E_n(A_1)).$$

Now let $\psi_\lambda : P_{\lambda\#}L \to L_\lambda$ be a natural map. Then

$$\phi_\lambda \oplus \psi_\lambda : P_{\lambda\#}M \oplus P_{\lambda\#}L \to M_\lambda \oplus L_\lambda$$

may be viewed as a natural map from $P_{\lambda\#}(M \oplus L)$ to $M_\lambda \oplus L_\lambda$ by setting $P_{\lambda\#}M \oplus P_{\lambda\#}L = P_{\lambda\#}(M \oplus L)$. From the assertion (I) it follows that $\phi_\lambda \oplus \psi_\lambda$ is an isomorphism, and hence ϕ_λ is also an isomorphism. This completes the proof of the proposition. □

3.9. Remark. Let $\bar{\Phi} : F_{1\#}P_1M \to F_{2\#}P_{2\#}M$ be a natural isomorphism. Then the fibre product $M_A(P_{1\#}M, \bar{\Phi}, P_{2\#}M)$ is well defined. From the definition of ϕ_λ in Proposition 3.8 the diagram

$$\begin{tikzcd}
M \arrow[r, "\bar{P}_\lambda"] \arrow[dr, "P'_\lambda"'] & M_\lambda \\
& P_{\lambda\#}M \arrow[u, "\phi_\lambda"']
\end{tikzcd}$$

is commutative, where P'_λ denotes the natural map. Therefore if we identify $P_{\lambda\#}M$ with M_λ by the isomorphism ϕ_λ, then it is easy to see that $\Phi = \bar{\Phi}$. This shows that $M = M_A(M_1, \Phi, M_2) = M_A(P_{1\#}M, \bar{\Phi}, P_{2\#}M)$.

3.10. Proposition. *Suppose $\{f_1, f_2\}$ is a Γ-pair. Then $S_A(M)$ is A-isomorphic to M^S.*

Proof. Since M is a finitely-generated projective A-module (Proposition 3.6), the component $S_A^r(M)$ of $S_A(M)$ in degree r is also a finitely-generated projective A-module. Thus $S_A^r(M)$ is naturally isomorphic to

$$N^{(r)} = M_A(P_{1\#}S_A^r(M), \Phi^{(r)}, P_{2\#}S_A^r(M)),$$

where $\Phi^{(r)}: F_{1\#}P_{1\#}S_A^r(M) \to F_{2\#}P_{2\#}S_A^r(M)$ denotes the natural isomorphism (Remark 3.9). By the isomorphism $\phi_\lambda : P_{\lambda\#}M \to M_\lambda$ as in Proposition 3.8, we may assume $P_{\lambda\#}M = M_\lambda$ and $P_{\lambda\#}S_A^r(M) = S_A^r(M_\lambda)$. Let

$$S(\Phi)_r : S_{\bar{A}}^r(F_{1\#}M_1) \to S_{\bar{A}}^r(F_{2\#}M_2)$$

be an isomorphism induced from $S(\Phi)$. Then it is easy to see that

$$N^{(r)} = M_A(S_{A_1}^r(M_1), S(\Phi)_r, S_{A_2}^r(M_2)),$$

and hence

$$M^S = \bigoplus_{r\geq 0} N^{(r)} \cong \bigoplus_{r\geq 0} S_A^r(M) = S_A(M).$$

which completes the proof. $\square$

3.11. Proposition. *Suppose $\{f_1, f_2\}$ is a Γ-pair. Then A is strongly projective over R.*

Proof. The proof consists of two steps.

(I) First we consider the special case where $A_\lambda = R_\lambda^{[n]}$ and $\alpha \in \mathrm{GA}_n(\bar{R})$ (3.2). Recall that $F_{\lambda\#}A_\lambda^n = \bar{A}^n$ (3.3) and $f_{\lambda\#}A = \bar{R}^{[n]} = \bar{A}$ (3.2). Thus the $\bar{A}$-module isomorphism $J(\alpha)^{-1}: F_{1\#}A_1^n \to F_{2\#}A_2^n$ is well defined because $J(\alpha)^{-1} \in \mathrm{GL}_n(\bar{R}^{[n]})$. Let us set $M_\lambda = A_\lambda^n$ and $\Phi = J(\alpha)^{-1}$. Then

$$M = M_A(M_1, \Phi, M_2) = M_A(A_1^n, J(\alpha)^{-1}, A_2^n).$$

If we identify $S_{\bar{A}}(\bar{A}^n)$ with $\bar{A}[X_{n+1}, \dots, X_{2n}] = \bar{R}^{[2n]}$ by an obvious isomorphism, then $S(J(\alpha)^{-1})$ may be considered as an element of $\mathrm{GA}_{2n}(\bar{R})$. From the definition of $F_1': A_1^n \to F_{1\#}A_1^n$ (3.3), it easily follows that $S(J(\alpha)^{-1} \circ F_1') = \pi_n(J(\alpha)^{-1})\alpha \circ \bar{f}_1 = \tilde{\alpha} \circ \bar{f}_1$, where $\bar{f}_1 : R_1^{[2n]} \to \bar{R}^{[2n]}$ is a natural map. So M^S is isomorphic to $A' = \mathrm{AL}_R(R_1^{[2n]}, \tilde{\alpha}, R_2^{[2n]})$ as an R-algebra. Since $S_A(M) \cong_A M^S$ (Proposition 3.10), in order to show that A is strongly projective over R, it suffices to prove that A' is strongly projective over R (Proposition 1.10). From Proposition 2.9(ii) we have $\tilde{\alpha} \in \mathrm{GA}_{2n}(\bar{R}_2)\mathrm{EA}_{2n}(\bar{R}_1)$. Thus $\tilde{\alpha}$ is of the form $\tilde{\alpha} = \alpha_2\alpha_1$ for $\alpha_2 \in \mathrm{GA}_{2n}(\bar{R}_2)$ and $\alpha_1 \in \mathrm{EA}_{2n}(\bar{R}_1)$. Since $\mathrm{EA}_{2n}(\bar{R}_1) = f_1^*(\mathrm{EA}_{2n}(R_1))$, we have $A' \cong_R \mathrm{AL}_R(R_1^{[2n]}, \alpha_2, R_2^{[2n]})$ (3.2). Replacing A by A', we may assume from the first that α is contained in $\mathrm{GA}_n(\bar{R}_2)$. So $\tilde{\alpha} \in f_2^*(\mathrm{GA}_{2n}(R_2))$ by Proposition 2.9(i), and therefore $A' \cong R^{[2n]}$ (3.2), which completes the proof of the assertion (I).

(II) Next we consider the general case. Let $M = M_A(M_1, \Phi, M_2)$ be as in Proposition 3.4. Then $M^S \cong_R \mathrm{AL}_R(R_1^{[n]}, \beta, R_2^{[n]})$ for some $\beta \in \mathrm{GA}_n(\bar{R})$. Thus M^S is strongly projective over R by the assertion (I), and hence $S_A(M)$ is also strongly projective over R because $S_A(M) \cong_R M^S$ by Proposition 3.10. This shows that A is strongly projective over R (Proposition 1.10), which completes the proof of the proposition. $\square$

3.12. Proposition. *Let B be a flat R-algebra. Suppose $\{f_1, f_2\}$ is a Γ-pair and suppose $p_{\lambda\#}B$ is a strongly projective R_λ-algebra for each $\lambda = 1, 2$. Then B is strongly projective over R.*

Proof. The diagram (1) induces the following pullback diagram of R-algebras:

$$\begin{tikzcd} B \arrow[r, "p_2'"] \arrow[d, "p_1'"'] & p_{2\#}B \arrow[d, "\bar{f}_2"] \\ p_{1\#}B \arrow[r, "\bar{f}_1"] & f_2 p_{2\#}B \end{tikzcd}$$

Since $f_1 \circ p_1 = f_2 \circ p_2$, there is a natural isomorphism $\beta : f_{1\#}p_{1\#}B \to f_{2\#}p_{2\#}B$ and the fibre product $\mathrm{AL}_R(p_{1\#}B, \beta, p_{2\#}B)$ is well defined. Furthermore $\mathrm{AL}_R(p_{1\#}B, \beta, p_{2\#}B)$ is strongly projective over R by Proposition 3.11. Therefore B is also strongly projective over R because $B \cong \mathrm{AL}_R(p_{1\#}, \beta, p_{2\#}B)$. $\square$

4. Quasi-polynomial algebras

4.1. Let A be an R-algebra. We will denote by $\tilde{R}$ the integral closure of R in the total quotient ring of R. Recall that A is called a quasi-polynomial algebra in n-variables if $j(\mathfrak{p}) \otimes_R A \cong j(\mathfrak{p})^{[n]}$ for each prime ideal $\mathfrak{p}$ of R, where $j(\mathfrak{p}) = R/\mathfrak{p}$. Let $f : R \to S$ be a ring homomorphism. Suppose $\mathfrak{p} \subset R$ and $P \subset S$ be a pair of prime ideals such that $\mathfrak{p} = f^{-1}(P)$. Then there is a natural ring homomorphism $j(\mathfrak{p}) \to j(P)$. Thus if A is a quasi-polynomial R-algebra, then $S \otimes_R A$ is a quasi-polynomial S-algebra.

4.2. Lemma. *Let R be a reduced noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra in n-variables. Then there is a finitely-generated R-subalgebra S of $\tilde{R}$ such that $S \otimes_R A \cong S^{[n]}$.*

Proof. Since R is reduced noetherian ring, R is the direct product $j(\mathfrak{p}_1) \times \cdots \times j(\mathfrak{p}_m)$ for all minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ of R. Furthermore $j(\mathfrak{p}_i) \otimes_R A \cong j(\mathfrak{p}_i)^{[n]}$ for $i = 1, \dots, m$, and hence $\tilde{R} \otimes_R A \cong \tilde{R}^{[n]}$. Thus we may assume that A is an R-subalgebra of $\tilde{R}^{[n]}$ such that $\tilde{R} \otimes_R A \cong \tilde{R}[A] = \tilde{R}^{[n]}$ because A is R-flat. Suppose $A = R[h_1, \dots, h_r]$ for $h_1, \dots, h_r \in \tilde{R}^{[n]}$. We can choose $F_i = F_i(X_1, \dots, X_r) \in \tilde{R}^{[r]}$ so that $F_i(h_1, \dots, h_r) = X_i$ $(i = 1, \dots, n)$. Let S be an R-subalgebra of $\tilde{R}$ generated by all coefficients of $h_1, \dots, h_r \in \tilde{R}^{[n]}$ and $F_1, \dots, F_n \in \tilde{R}^{[r]}$. Then $S \otimes_R A \cong S[A] = S^{[n]}$ as required. $\square$

4.3. Lemma. *Let R be a noetherian ring with the nil-radical $\mathfrak{r}$, and let A be a flat R-algebra. Suppose $R/\mathfrak{r} \otimes_R A \cong (R/\mathfrak{r})^{[n]}$. Then $A \cong R^{[n]}$.*

Proof. We can choose $h_1, \dots, h_n \in A$ so that $A = R[h_1, \dots h_n] + \mathfrak{r}A$ and the residue classes $\bar{h}_i \equiv h_i \pmod{\mathfrak{r}A}$ $(i = 1, \dots, n)$ are algebraically independent over $R/\mathfrak{r}$. Since $\mathfrak{r}$ is finitely-generated, we get a positive integer r such that $\mathfrak{r}^r = (0)$. Therefore $A = R[h_1, \dots, h_n] + \mathfrak{r}A = R[h_1, \dots, h_n]$. Now let $F(X_1, \dots, X_n)$ be an element of $R^{[n]}$

such that $F(h_1,\dots,h_n)\in\mathfrak{r}[h_1,\dots,h_n]$. Then $F(X_1,\dots,X_n)\in\mathfrak{r}[X_1,\dots,X_n]$ because $\bar{h}_i$ are algebraically independent over $R/\mathfrak{r}$. This shows that the kernel $\mathfrak{a}$ of R-homomorphism $\phi: R^{[n]}\to R[h_1,\dots,h]$ defined by $\phi(X_i)=h_i$ is contained in $\mathfrak{r}[X_1,\dots,X_n]$. Since $A=R[h_1,\dots,h_n]$ is R-flat, we have $\mathfrak{a}=(0)$ by [17, p. 445], which completes the proof. □

4.4. Lemma. *Let R be a noetherian ring with the nil-radical $\mathfrak{r}$, and let $\bar{R}=R/\mathfrak{r}$. Suppose a flat R-algebra A satisfies the following two conditions:*

(i) *$\Omega_R(A)$ is a finitely-generated projective A-module.*

(ii) *$\bar{A}=\bar{R}\otimes_R A$ is strongly projective over $\bar{R}$.*

Then A is strongly projective over R.

Proof. We can choose some projective A-module Q so that $\Omega_R(A)\oplus Q\cong A^n$. From the hypothesis (ii) we can find a projective $\bar{A}$-module $\bar{M}$ so that $S_{\bar{A}}(\bar{M})\cong_{\bar{R}}\bar{R}^{[m]}$ for some m. Note that $\Omega_{\bar{R}}(\bar{A})\oplus\bar{M}\cong\bar{A}^m$ (Corollary 1.9). If we define $\bar{Q}=\bar{A}\otimes_A Q$, then

$$\bar{M}\oplus\bar{A}^n\cong\bar{M}\oplus\Omega_{\bar{R}}(\bar{A})\oplus\bar{Q}\cong\bar{A}^m\oplus\bar{Q}.$$

Therefore

$$\bar{R}\otimes_R S_A(A^m\oplus Q)\cong_{\bar{A}} S_{\bar{A}}(\bar{M}\oplus\bar{A}^n)\cong_{\bar{R}}\bar{R}^{[n+m]}.$$

On the other hand, $S_A(A^m\oplus Q)$ is A-flat, and hence $S_A(A^m\oplus Q)$ is R-flat because A is R-flat. Thus $S_A(A^m\oplus Q)\cong_R R^{[m+n]}$ by Lemma 4.3. This shows that A is strongly projective over R. □

4.5. Theorem. *Let R be a noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra. Suppose $\Omega_R(A)$ is a projective A-module. Then A is a strongly projective R-algebra.*

Proof. By virtue of Lemma 4.4 we may assume that R is reduced. Suppose A is not strongly projective over R. Let S be a finitely-generated R-subalgebra of $\bar{R}$ such that $S\otimes_R A\cong S^{[n]}$ as in Lemma 4.2. There is a non-zero divisor c in S such that $cS\subset R$. If c is a unit in S, then $S=R$, and hence $A\cong R^{[n]}$. This contradicts the assumption that A is not strongly projective over R. Therefore cS is a proper ideal of R. Now consider the following commutative diagram of rings:

$$\begin{tikzcd} R \arrow[r, "g_1"] \arrow[d, "g_2"'] & S \arrow[d, "f_2"] \\ R/cS \arrow[r, "f_1"] & S/cS \end{tikzcd}$$

where g_1, f_1 are the natural injections and g_2, f_2 are the canonical projections. This diagram is clearly a pullback diagram and $\{f_1, f_2\}$ is a Γ-pair. If $R/cs\otimes_R A$ is strongly projective over R/cs, then A is strongly projective over R by Proposition

3.12, which is a contradiction. Thus $R/cs\otimes_R A$ is not strongly projective over R/cs. Let $\sqrt{cS}$ denote the radical of cS in R, and let $R_{(1)}=R/\sqrt{cS}$. Then $R_{(1)}\otimes_R A$ is not strongly projective over $R_{(1)}$ (Lemma 4.4). Similarly we can construct a reduced noetherian ring $R_{(2)}=R_{(1)}/\mathfrak{a}$ for some proper radical ideal $\mathfrak{a}$ of $R_{(1)}$ such that $R_{(2)}\otimes_R A$ is not strongly projective over $R_{(2)}$. If we continue this process, then we obtain an infinite sequence

$$R \xrightarrow{\phi_0} R_{(1)} \xrightarrow{\phi_1} R_{(2)} \longrightarrow \cdots \longrightarrow R_{(i)} \xrightarrow{\phi_i} \cdots$$

of canonical projections ϕ_i such that Ker $\phi_i \neq (0)$ $(i=0,1,2,\ldots)$. This contradicts the hypothesis that R is noetherian. Therefore A is strongly projective over R, which completes the proof. □

4.6. Remark. Let R be a reduced noetherian ring such that $\tilde{R}$ is a finitely-generated R-algebra, and let A be a quasi-polynomial R-algebra. Then it follows from lemma 4.2 that $\tilde{R}\otimes_R A\cong \tilde{R}^{[n]}$. Thus A is necessarily a finitely-generated flat R-algebra with the differential A-module $\Omega_R(A)$ projective, [13], [19].

4.7. Corollary. *Let R be a noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra. Then A is stably equivalent to $R^{[n]}$ if and only if $\Omega_R(A)$ is stably equivalent to A^n.*

Proof. Suppose $\Omega_R(A)$ is stably equivalent to A^n. Then $\Omega_R(A)\oplus A^r\cong A^{n+r}$ for some r. Since A is strongly projective over R by Theorem 4.5, we can choose a projective A-module M so that $S_A(M)\cong R^{[m]}$. Note that $\Omega_R(A)\oplus M\cong A^m$ by Corollary 1.9. Therefore $A^{n+r}\oplus M\cong A^{m+r}$, and hence $A^{[m+r]}\cong_A S_A(M)^{[n+r]}\cong_R R^{[m+n+r]}$. This shows that A is stably equivalent to $R^{[n]}$. Thus we complete the proof of the 'if' part. The proof of the 'only if' part is obvious. □

5. Algebras with polynomial fibres

5.1. Lemma. *Let R be a reduced noetherian ring, and let A be a finitely-generated flat R-algebra. Suppose $k(\mathfrak{p})\otimes_R A\cong k(\mathfrak{p})^{[n]}$ for each prime ideal $\mathfrak{p}$ of R. Then $\Omega_R(A)$ is projective over A.*

Proof. The total quotient ring K of R is the direct product $k(\mathfrak{p}_1)\times\cdots\times k(\mathfrak{p}_r)$ for all minimal prime ideals $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ of R. Thus $K\otimes_R A\cong K^{[n]}$. Since A is R-flat, we may assume that A is an R-subalgebra of $K^{[n]}$ such that $K\otimes_R A=K[A]=K^{[n]}$. Let P be a prime ideal of A. Suppose $\mathfrak{p}_1\subset P,\ldots,\mathfrak{p}_m\subset P, \mathfrak{p}_{m+1}\not\subset P,\ldots,\mathfrak{p}_r\not\subset P$. If we denote by L the total quotient ring of A_P, then L is the direct product $k(\mathfrak{p}_1A)\times\cdots\times k(\mathfrak{p}_mA)$ because $\mathfrak{p}_iA$ is the minimal prime ideal of A such that $\mathfrak{p}_iA\subset P$ for each $i=1,\ldots,m$. Let us set $\mathcal{L}=(k(\mathfrak{p}_1)\otimes_R A)\times\cdots\times(k(\mathfrak{p}_m)\otimes_R A)$. There is an injection $\mathcal{L}\hookrightarrow L$. Furthermore from the hypothesis that $k(\mathfrak{p}_i)\otimes_R A\cong k(\mathfrak{p}_i)^{[n]}$ $(i=1,\ldots,m)$ we have $\mathcal{L}\otimes_A\Omega_R(A)\cong\mathcal{L}^n$. This shows that $L\otimes_A \Omega_R(A)\cong L\otimes_{\bar L}(\bar L\otimes_A \Omega_R(A))\cong L^n$, and hence $L\otimes_{A_P}(A_P\otimes_A \Omega_R(A))\cong L^n$. On the other hand, for a prime ideal $\mathfrak{p}=P\cap R$ of R, there is a natural map $k(\mathfrak{p})\otimes_R A\to k(P)$. Thus $k(P)\otimes_{A_P}(A_P\otimes_A \Omega_R(A))\cong k(P)^n$ because $k(\mathfrak{p})\otimes_R \Omega_R(A)\cong (k(\mathfrak{p})\otimes_R A)^n$. It follows from Nakayama's Lemma that $A_P\otimes_A \Omega_R(A)$ is free over A_P. So $\Omega_R(A)$ is projective, which completes the proof. □

5.2. Corollary. *Let R be a reduced noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra. Then A is strongly projective over R.*

Proof. This is an immediate consequence of Theorem 4.5 and Lemma 5.1. □

5.3. Lemma. *Let R be a local ring, and let $f:R\to\bar R$ be a ring homomorphism. Suppose A is a finitely-generated augmented R-algebra satisfying the following two conditions:*

(i) *$\Omega_R(A)$ is projective over A.*

(ii) *$\bar R\otimes_R A\cong_R S_{\bar R}(M)$ for some projective $\bar R$-module M. Then $\bar R\otimes_R A\cong \bar R^{[n]}$ for some n.*

Proof. It suffices to show that M is free. We may identify $\bar R\otimes_R A$ with $S_{\bar R}(M)$ by the isomorphism in (ii). Let $g:A\to R$ be an augmentation. Then we get a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f'"'] & R \arrow[d, "f"] \\ f_\# A=\bar R\otimes_R A \arrow[r, "g'"] & \bar R \end{tikzcd}$$

of rings, where f' and g' are the natural maps. Thus $g'_\# f'_\# \Omega_R(A)\cong f_\# g_\# \Omega_R(A)$. Since $\bar R\otimes_R A=S_{\bar R}(M)$, we have

$$f'_\#\Omega_R(A)\cong \Omega_{\bar R}(f_\# A)\cong S_{\bar R}(M)\otimes_{\bar R} M.$$

Therefore

$$f_\# g_\# \Omega_R(A)\cong g'_\#(S_{\bar R}(M)\otimes_{\bar R} M)\cong M.$$

Note that $g_\#\Omega_R(A)$ is a free R-module, and hence M is a free $\bar R$-module, which completes the proof. □

5.4. Let t be a non-unit element of a ring R. We denote by $\hat R$ the tR-adic completion of R.

5.5. Lemma. *Let $R_r=R/t^rR$, and let A be an R-algebra such that $R_r\otimes_R A$ is weakly projective over R_r for each positive integer r. Then $\hat R\otimes_R A$ is an augmented $\hat R$-algebra.*

Proof. Let $\phi_r : R_{r+1} \to R_r$ be a canonical projection, and let $\phi_r' : R_{r+1} \otimes_R A \to R_r \otimes_R A$ be an induced homomorphism. Since $R_r \otimes_R A$ is weakly projective over R_r, there is an augmentation $f_r : R_r \otimes_R A \to R_r$. From the definition of weakly projective algebra, [8, p. 283], we can choose an augmentation $f_{r+1} : R_{r+1} \otimes_R A \to R_{r+1}$ so that $\phi_r \circ f_{r+1} = f_r \circ \phi_r'$. If $g_r : R_r \to R_r \otimes_R A$ is a natural map, then $f_r \circ g_r$ is the identity map on R_r. Now the sets $\{g_r\}$ and $\{f_r\}$ give rise to the following homomorphisms

$$\varprojlim R_r \xrightarrow{g_\infty} \varprojlim (R_r \otimes_R A) \xrightarrow{f_\infty} \varprojlim R_r$$

of inverse limits such that $f_\infty \circ g_\infty$ is the identity map on $\varprojlim R_r$. In other words, f_∞ is an augmentation of an $\hat{R}$-algebra $\hat{A}$, where $\hat{A}$ denotes the tA-adic completion of A. If $\psi : \hat{R} \otimes_R A \to \hat{A}$ is a natural map, then $f_\infty \circ \psi : \hat{R} \otimes_R A \to \hat{R}$ is clearly an augmentation, and hence $\hat{R} \otimes_R A$ is an augmented $\hat{R}$-algebra. $\square$

5.6. An R-algebra A is called a locally quasi-polynomial R-algebra in n-variables if $R_\mathfrak{p} \otimes_R A$ is a quasi-polynomial $R_\mathfrak{p}$-algebra in n-variables for each prime ideal $\mathfrak{p}$ of R.

5.7. Theorem. *Let R be a noetherian ring, and let A be a finitely-generated flat R-algebra. Suppose $k(\mathfrak{p}) \otimes_R A \cong k(\mathfrak{p})^{[1]}$ for each prime ideal $\mathfrak{p}$ of R. Then A is a locally quasi-polynomial R-algebra in one variable.*

Proof. From the definition of locally quasi-polynomial algebras, we may assume that R is an integral local domain. The proof consists of two steps.

(I) First we consider the special case where A is an augmented R-algebra. In order to prove Theorem 5.7 it suffices to show that $\tilde{R} \otimes_R A \cong \tilde{R}[X]$ $(\cong \tilde{R}^{[1]})$. Let K be a quotient field of R. We may assume that A is an R-subalgebra of $K[X]$ such that $K \otimes_R A = K[A] = K[X]$. Since A is finitely-generated over R, we can take an element $t \in R$ so that $R[\frac{1}{t}][A] = R[\frac{1}{t}][X]$. If t is a unit element in R, then $A = R[X]$, and hence A is a locally quasi-polynomial algebra. Thus we may assume that t is a non-unit element in R. Given a prime ideal P of $\tilde{R}$, the localization $\bar{R} = \tilde{R}_P$ is a Krull ring, and therefore $\bar{R}$ is of the form

$$\bar{R} = \bar{R}[\tfrac{1}{t}] \cap \Big(\bigcap_{v \in V} R_v \Big),$$

where V is a finite set of discrete valuations v of K whose valuation rings R_v contain $\bar{R}$. Note that $\bar{R}[\frac{1}{t}] \otimes_R A = \bar{R}[\frac{1}{t}][A]$ and $R_v \otimes_R A = R_v[A]$. Therefore

$$\bar{R}[A] = \bar{R} \otimes_R A = \bar{R}[\tfrac{1}{t}][A] \cap \Big(\bigcap_{v \in V} R_v[A] \Big).$$

By virtue of [15] we have $R_v[A] \cong R_v[X]$. Thus $R_v[A]$ is of the form $R_v[A] = R_v[a_v X + b_v]$ for $a_v \in K - (0)$ and $b_v \in K$. Now let $g : A \to R$ be an augmentation, and let $g' : K \otimes_R A \to K$ be an augmentation induced from g. Then $g'(\bar{R}[\frac{1}{t}][A]) = \bar{R}[\frac{1}{t}]$ and $g'(R_v[A]) = R_v$. This shows that $g'(X) \in \bar{R}[\frac{1}{t}]$ and $a_v g'(X) + b_v \in R_v$. If we set

$Y = X - g'(X)$, then $R[\frac{1}{t}][X] = \bar{R}[\frac{1}{t}][Y]$ and $R_v[a_v X + b_v] = R_v[a_v Y]$. So we get

$$\bar{R}[A] = \bar{R}[\tfrac{1}{t}][Y] \cap \Big(\bigcap_{v \in V} R_v[a_v Y]\Big) = \bigoplus_{j \geq 0} M_j Y^j,$$

where

$$M_j = \bar{R}[\tfrac{1}{t}] \cap \Big(\bigcap_{v \in V} R_v a_v^j\Big).$$

Let $A = R[h_1, \ldots, h_n]$. Then

$$\bar{R}[A] = \bar{R}[h_1, \ldots h_n] = \bigoplus_{j \geq 0} M_j Y^j.$$

Thus h_i is of the form $h_i = m_{i1} Y + \cdots + m_{ir} Y^r$ for $m_{ij} \in M_j$ ($j = 1, \ldots, r$, $i = 1, \ldots, n$), where r is the maximal Y-degree of $h_1, \ldots, h_n$. Let us set $N_j = \bar{R} m_{1j} + \cdots + \bar{R} m_{nj} \subset M_j$ ($j = 1, \ldots, r$), the $\bar{R}$-submodule of M_j generated by $m_{1j}, \ldots, m_{nj}$. It is easy to see that each N_j is a fractional ideal of $\bar{R}$ such that $\bar{R}[A] = \bar{R}[N_1 Y, \ldots, N_r Y^r]$. Recall that $\bar{R}[A] = \bar{R} \otimes_R A$ is $\bar{R}$-flat, and hence N_1 is a projective $\bar{R}$-module because N_1 is a direct summand of $\bar{R}[A]$. Since $\bar{R}$ is local, N_1 is a free $\bar{R}$-module of rank one. In other words, $\bar{R}[A]$ is of the form $\bar{R}[A] = \bar{R}[aY, N_2 Y^2, \ldots, N_r Y^r]$ for some $a \in K$. Therefore

$$R_v[A] = R_v[a_v Y] = R_v[aY, N_2 Y^2, \ldots, N_r Y^r]$$

and

$$\bar{R}[\tfrac{1}{t}][A] = \bar{R}[\tfrac{1}{t}][Y] = \bar{R}[\tfrac{1}{t}][aY, N_2 Y^2, \ldots, N_r Y^r],$$

which imply $R_v[a_v Y_v] = R_v[aY]$ and $\bar{R}[\frac{1}{t}][Y] = \bar{R}[\frac{1}{t}][aY]$. This shows that

$$\bar{R}[A] = \bar{R}[\tfrac{1}{t}][aY] \cap \Big(\bigcap_{v \in V} R_v[aY]\Big) = \bar{R}[aY] \cong \bar{R}[X],$$

i.e. $\tilde{R} \otimes_R A = \tilde{R}[A]$ is a locally polynomial $\tilde{R}$-algebra in one variable. According to [4], we can choose a projective $\tilde{R}$-module M of rank one so that $\tilde{R} \otimes_R A \cong S_{\tilde{R}}(M)$. Note that $\Omega_R(A)$ is projective over A by Lemma 5.1; so it follows from Lemma 5.2 that $\tilde{R} \otimes_R A \cong \tilde{R}[X]$ as required.

(II) Next we consider the general case by induction on (Krull) dimension $\dim R = r$. If $r = 0$, then R is a field and Theorem 5.7 is obvious. Suppose $\dim R = r$. As shown in the proof of (I) we can take $t \in R$ so that

$$R[\tfrac{1}{t}][A] = R[\tfrac{1}{t}][X] \subset K[A] = K[X],$$

where K is a quotient field of R. We may assume t a non-unit element in R. From the induction hypothesis, $R_r \otimes_R A$ is a quasi-polynomial R_r-algebra in one variable, where $R_r = R/t^r R$ for any positive integer r. On the other hand, $\Omega_R(A)$ is a projective A-module by Lemma 5.1, and therefore $\Omega_{R_r}(R_r \otimes_R A)$ is projective over $R_r \otimes_R A$. This shows that $R_r \otimes_R A$ is strongly projective over R_r by Theorem 4.5. A strongly projective algebra is weakly projective (1.3). Thus $\hat{R} \otimes_R A$ is an augmented $\hat{R}$-algebra (Lemma 5.5). Furthermore $\hat{R} \otimes_R A$ is a finitely-generated flat $\hat{R}$-algebra such that $k(P) \otimes_R A \cong k(P)[X]$ for each prime ideal P of $\hat{R}$. Note that $\hat{R}$ is a noetherian local ring and, by the assertion (I) above, $\hat{R}\otimes_R A$ is a quasi-polynomial $\hat{R}$-algebra. This shows that $\hat{R}\otimes_R A$ is strongly projective over $\hat{R}$ (Theorem 4.5). Now consider the following commutative diagram of rings:

$$\begin{array}{ccc} R & \xrightarrow{g_1} & \hat{R} \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ R\left[\frac{1}{t}\right] & \xrightarrow{f_1} & \hat{R}\left[\frac{1}{t}\right] \end{array} \tag{8}$$

where g_λ, f_λ $(\lambda=1,2)$ are natural injections. Since $\hat{R}$ is faithfully flat over R, we have $R=R\left[\frac{1}{t}\right]\cap\hat{R}$, i.e. the diagram (8) is a pullback diagram. Let $R[[X]]$ be a formal power series ring in one variable. Then $\hat{R}=R[[X]]/(X-t)$, and hence each element $a\in\hat{R}\left[\frac{1}{t}\right]$ is of the form $a=a_0t^k+a_1t^{k+1}+\cdots$ for some integer k. Thus $\hat{R}\left[\frac{1}{t}\right]$ is of the form $\hat{R}\left[\frac{1}{t}\right]=R\left[\frac{1}{t}\right]+t\hat{R}$. This shows that $\{f_1,f_2\}$ in the diagram (8) is a Γ-pair, and therefore A is strongly projective over R by Proposition 3.12. In particular A is an augmented R-algebra. From the assertion (I) it follows that A is a quasi-polynomial R-algebra, which completes the proof of the theorem. □

5.8. Corollary. *Let R be a reduced noetherian ring. Then the following two conditions are equivalent:*

(i) *A is a finitely-generated, flat, locally, quasi-polynomial R-algebra in one variable.*

(ii) *A is a weakly projective R-algebra such that* $\dim k(\mathfrak{p})\otimes_R A=1$ *for each prime ideal $\mathfrak{p}$ of R.*

Proof. (i) ⇒ (ii): This implication follows from Theorem 5.7 and Corollary 5.2.

(ii) ⇒ (i): Suppose A is a weakly projective R-algebra. Then A is clearly finitely-generated flat over R. Furthermore $k(\mathfrak{p})\otimes_R A$ is weakly projective over $k(\mathfrak{p})$. Thus $k(\mathfrak{p})\otimes_R A\cong k(\mathfrak{p})^{[1]}$, [8], [10]. □

6. Invertible algebras

6.1. Let R be a ring. An R-algebra A is called invertible if there is an R-algebra B such that $A\otimes_R B\cong_R R^{[n]}$ for some n. As for the basic properties of invertible algebras we refer to [8]. (Called 'projective' algebras in [8].)

6.2. Proposition. *Let R be a local ring, and let A be an invertible R-algebra. Then $\Omega_R(A)$ is stably equivalent to a free A-module of finite rank.* (See [14, p. 330].)

Proof. Let B be an invertible R-algebra such that $A\otimes_R B\cong R^{[n]}$. An invertible algebra is weakly projective, [8]. Let $f: B\to R$ be an augmentation. Then f induces an augmentation $f':A\otimes_R B\to A$ as an A-algebra. Since $\Omega_R(A\otimes_R B)\cong(A\otimes_R B)^n$, we have

$$(A\otimes_R B)^n\cong\Omega_R(A)\otimes_R B\oplus A\otimes_R\Omega_R(B).$$

Therefore we obtain an A-isomorphism

$$f'_{\#}(A\otimes_R B)^n\cong f'_{\#}(\Omega_R(A)\otimes_R B)\oplus f'_{\#}(A\otimes_R\Omega_R(B)).$$

Note that $f'_{\#}(A\otimes_R B)^n\cong A^n$ and $f'_{\#}(\Omega_R(A)\otimes_R B)\cong\Omega_R(A)$. On the other hand, $f'_{\#}(A\otimes_R\Omega_R(B))\cong A\otimes_R f_{\#}\Omega_R(B)$. This shows that $f'_{\#}(A\otimes_R\Omega_R(B))$ is a free A-module because $f_{\#}\Omega_R(B)$ is a free R-module. Thus $\Omega_R(A)$ is stably equivalent to a free A-module of finite rank. □

6.3. Lemma. *Let $R\subset S$ be a ring extension, and let M be a finitely-generated projective $R^{[n]}$-module. Suppose $S\otimes_R M$ is extended from S. Then there is a finitely-generated R-subalgebra $\bar{R}$ of S such that $\bar{R}\otimes_R M$ is extended from $\bar{R}$.*

Proof. We can choose an $R^{[n]}$-module M' so that $M\oplus M'\cong R^{[n]r}$ for some r. We may assume $M\oplus M'=R^{[n]}e_1\oplus\cdots\oplus R^{[n]}e_r$, where $\{e_1,\dots,e_r\}$ is a basis of $R^{[n]r}$. Suppose M is generated by $F_1,\dots,F_r\in R^{[n]r}$. Then $S\otimes_R M\cong S^{[n]}F_1+\cdots+S^{[n]}F_r$, where the right hand side is considered as an $S^{[n]}$-submodule of $S^{[n]r}=S^{[n]}e_1\oplus\cdots\oplus S^{[n]}e_r$. Since $S\otimes_R M$ is extended from S, we can find a finitely-generated projective S-module N so that $S\otimes_R M\cong S^{[n]}\otimes_S N$. Let N' be an S-module such that $N\oplus N'\cong S^m$ for some m. We may assume $N\oplus N'=S^m$. Suppose $N=SG_1+\cdots+SG_m$ and $N'=SG'_1+\cdots+SG'_m$ for $G_i,G'_i\in S^m$. Then $S^{[n]}\otimes_S N\cong S^{[n]}G_1+\cdots+S^{[n]}G_m$. Thus there is an $S^{[n]}$-isomorphism

$$\phi:S^{[n]}G_1+\cdots+S^{[n]}G_m\to S^{[n]}F_1+\cdots+S^{[n]}F_r.$$

Now it is easy to see that there is a finitely-generated R-subalgebra $\bar{R}$ of S satisfying the following two conditions:

(a) $\bar{R}^{[n]}\phi(G_1)+\cdots+\bar{R}^{[n]}\phi(G_m)=\bar{R}^{[n]}F_1+\cdots+\bar{R}^{[n]}F_r$,

(b) $(\bar{R}G_1+\cdots+\bar{R}G_m)\oplus(\bar{R}G'_1+\cdots+\bar{R}G'_m)=\bar{R}^m$.

This shows that

$$\bar{R}\otimes_R M\cong\bar{R}^{[n]}F_1+\cdots+\bar{R}^{[n]}F_r\cong\bar{R}^{[n]}G_1+\cdots+\bar{R}^{[n]}G_m$$
$$\cong\bar{R}^{[n]}\otimes_R(\bar{R}G_1+\cdots+\bar{R}G_m),$$

i.e. $\bar{R}\otimes_R M$ is extended from $\bar{R}$. □

6.4. Corollary. *Let $R\subset S$ be a ring extension, and let A be a weakly projective R-algebra. Let M be a finitely-generated projective A-module. Suppose $S\otimes_R M$ is extended from S. Then there is a finitely-generated R-subalgebra $\bar{R}$ of S such that $\bar{R}\otimes_R M$ is extended from $\bar{R}$.*

Proof. Since $S \otimes_R M$ is extended from S, the $S^{[n]}$-module $S^{[n]} \otimes_A M$ is extended from S. Thus there is a finitely-generated R-subalgebra $\bar{R}$ of S such that $\bar{R}^{[n]} \otimes_A M$ is extended from $\bar{R}$, i.e. $\bar{R}^{[n]} \otimes_A M \cong \bar{R}^{[n]} \otimes_{\bar{R}} \bar{M}$ for some $\bar{R}$-module $\bar{M}$. Now let $h: R^{[n]} \to A$ be an augmentation, and let $h': \bar{R}^{[n]} \to \bar{R} \otimes_R A$ be an induced augmentation. Then we have $h'_{\#}(\bar{R}^{[n]} \otimes_A M) \cong h'_{\#}(\bar{R}^{[n]} \otimes_{\bar{R}} \bar{M})$ as $\bar{R} \otimes_R A$-modules. From the definition of h', we see that $h'_{\#}(\bar{R}^{[n]} \otimes_A M) \cong \bar{R} \otimes_R M$ and $h'_{\#}(\bar{R}^{[n]} \otimes_{\bar{R}} \bar{M}) \cong (\bar{R} \otimes_R A) \otimes_{\bar{R}} \bar{M}$. Therefore we get $\bar{R} \otimes_R M \cong (\bar{R} \otimes_R A) \otimes_{\bar{R}} \bar{M}$. This shows that $\bar{R} \otimes_R M$ is extended from $\bar{R}$, which completes the proof of the corollary. □

The following lemma is due to Connell [8].

6.5. Lemma. *Let R be a ring, and let A be a weakly projective R-algebra. Then $\Omega_R(A)$ is finitely-generated projective over A.*

Proof. (See [8, p. 289].) Since A is weakly projective, we have a pair of ring homomorphisms $g: A \to R^{[n]}$ and $f: R^{[n]} \to A$ such that $f \circ g = \mathrm{id}_A$. Therefore we get a map $g_{\#}\Omega_R(A) \to \Omega_R(R^{[n]})$, which induces a map $F: f_{\#}g_{\#}\Omega_R(R) \to f_{\#}\Omega_R(R^{[n]})$. From the assumption $f \circ g = \mathrm{id}_A$ it follows that $f_{\#}g_{\#}\Omega_R(A)$ and $\Omega_R(A)$ are naturally isomorphic, and hence we may identify $\Omega_R(A)$ with $f_{\#}g_{\#}\Omega_R(A)$. On the other hand, g induces a map $G: f_{\#}\Omega_R(R^{[n]}) \to \Omega_R(A)$. It is easy to verify that $G \circ F$ is the identity on $\Omega_R(A)$. This shows that $\Omega_R(A)$ is a direct summand of a free A-module $f_{\#}\Omega_R(R^{[n]})$, which completes the proof. □

6.6. Lemma. *Let R be a noetherian ring with* $\dim R = r$, *and let A be an R-algebra stably equivalent to $R^{[1]}$. Then $A^{[n]} \cong R^{[n+1]}$ for $n = 2^r - 1$.*

Proof. Let $\mathfrak{r}$ be a nil-radical of R, and let $\bar{R} = R/\mathfrak{r}$. Then $\bar{R} \otimes_R A^{[n]} \cong \bar{R}^{[n+1]}$ for $n = 2^r - 1$, [1, p. 474]. Since A is R-flat, we have $A^{[n]} \cong R^{[n+1]}$ by Lemma 4.3. □

6.7. Theorem. *Let R be a ring, and let A be an R-algebra. Then the following two conditions are equivalent:*

(i) *A is an invertible R-algebra such that $\Omega_R(A) \in \mathrm{Pic}(A)$.*

(ii) *A is stably equivalent to $S_R(M)$ for some $M \in \mathrm{Pic}(R)$. (See* [14, p. 336].)

Proof. (i) ⇒ (ii): Let B be an R-algebra such that $A \otimes_R B \cong_R R^{[n]}$. Since the natural maps $A \to A \otimes_R B$ and $B \to A \otimes_R B$ are injective, [8], we may assume that A and B are R-subalgebras of $R^{[n]}$ such that $A \otimes_R B \cong A[B] = R^{[n]}$, i.e. $A \otimes_R B$ is naturally isomorphic to $R^{[n]}$. Suppose $A = R[F_1, \dots, F_n]$ and $B = R[G_1, \dots, G_n]$. Then we can choose a finitely-generated π-subalgebra $\bar{R}$ of R so that $\bar{R}[F_1, \dots, F_n, G_1, \dots, G_n] = \bar{R}^{[n]}$, where π denotes the prime ring of R. Note that $\bar{R}$ is a finite-dimensional noetherian ring. Let us set $\bar{A} = \bar{R}[F_1, \dots, F_n]$ and $\bar{B} = \bar{R}[G_1, \dots, G_n]$. Let $g: R^{[n]} \to R$ be an augmentation defined by $g(X_i) = 0$ $(i = 1, \dots, n)$, and let $h = g \mid B$, the restriction of g to B. Thus $h: B \to R$ is an augmentation such that $h(\bar{B}) = \bar{R}$. Consider the augmentation

$h': A \otimes_R B \to A$ induced from $h: B \to R$. Through the natural isomorphism $A \otimes_R B \to A[B]$, we may regard h' as an augmentation of $R^{[n]}$ such that $h'(\bar{B}) = \bar{R}$. Therefore we have $h'(\bar{R}^{[n]}) = h'(\bar{A}[B]) = \bar{A}$, and hence $\bar{A}$ is weakly projective over $\bar{R}$. Thus $\Omega_{\bar{R}}(\bar{A})$ is a finitely-generated projective $\bar{A}$-module (Lemma 6.5). Since $\Omega_R(A)$ is an element of $\mathrm{Pic}(A)$ such that $R_{\mathfrak{p}} \otimes_R \Omega_R(A)$ is stably equivalent to a free $A_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p}$ of R (Proposition 6.2), $R_{\mathfrak{p}} \otimes_R \Omega_R(A)$ itself is a free $A_{\mathfrak{p}}$-module of rank one. This shows that $\Omega_R(A)$ is extended from R (Proposition 1.7). Note that $R \otimes_{\bar{R}} \bar{A} \cong A$ and $R \otimes_{\bar{R}} \Omega_{\bar{R}}(\bar{A}) \cong \Omega_R(A)$. By virtue of Corollary 6.4 we can find a finitely-generated $\bar{R}$-subalgebra R' of R so that $R' \otimes_{\bar{R}} \Omega_{\bar{R}}(\bar{A})$ is extended from R'. Since $R' \otimes_{\bar{R}} \bar{A} \cong R'[\bar{A}]$, the $R'[\bar{A}]$-module $\Omega_{R'}(R'[\bar{A}])$ is extended from R'. Replacing $\bar{R}$ by R', we may assume from the first that $\Omega_{\bar{R}}(\bar{A})$ is extended from $\bar{R}$. Let P be any prime ideal of $\bar{R}$. Then $\bar{R}_P \otimes_{\bar{R}} \Omega_{\bar{R}}(\bar{A})$ is a free $\bar{A}_P$-module of rank one, which implies $k(P) \otimes_{\bar{R}} \Omega_{\bar{R}}(\bar{A}) \cong (k(P) \otimes_{\bar{R}} \bar{A})^1$, a free module of rank one over $k(P) \otimes_{\bar{R}} \bar{A}$. Thus $k(P) \otimes_{\bar{R}} \bar{A}$ is a weakly projective $k(P)$-algebra with $\dim k(P) \otimes_{\bar{R}} \bar{A} = 1$, [8, p. 289], and therefore we have $k(P) \otimes_{\bar{R}} \bar{A} \cong k(P)^{[1]}$ (see [8], [10]). It follows from Theorem 5.7 that $\bar{A}$ is locally quasi-polynomial $\bar{R}$-algebra in one variable. Since $\Omega_{\bar{R}}(\bar{A}_P)$ is a free $\bar{A}_P$-module of rank one, $\bar{A}_P$ is stably equivalent to $\bar{R}_P^{[1]}$ as an $\bar{R}_P$-algebra (Corollary 4.7). So $\bar{A}_P^{[n]} \cong \bar{R}_P^{[n+1]}$ for $n = 2^r - 1$, where $r = \dim \bar{R}$ (Lemma 6.6). In other words, $\bar{A}^{[n]}$ is locally polynomial $\bar{R}$-algebra, and therefore $\bar{A}^{[n]} \cong_{\bar{R}} S_{\bar{R}}(\bar{N})$ for some projective $\bar{R}$-module $\bar{N}$, [4]. Under this isomorphism we identify $\bar{A}^{[n]}$ with $S_{\bar{R}}(\bar{N})$. Let $\phi: S_{\bar{R}}(\bar{N}) \to \bar{R}$ be a natural augmentation, and let $\psi: \bar{A} \to \bar{R}$ be a restriction of ϕ to $\bar{A}$. Since $\Omega_{\bar{R}}(\bar{A}^{[n]}) \cong \bar{A}^{[n]} \otimes_A (\Omega_{\bar{R}}(\bar{A}) \oplus \bar{A}^n)$ and $\Omega_{\bar{R}}(S_{\bar{R}}(\bar{N})) \cong S_{\bar{R}}(\bar{N}) \otimes_{\bar{R}} \bar{N}$, we have $\psi_{\#}(\Omega_{\bar{R}}(\bar{A}) \oplus \bar{A}^n) \cong \phi_{\#}(S_{\bar{R}}(\bar{N}) \otimes_{\bar{R}} \bar{N})$, which implies $\psi_{\#}\Omega_{\bar{R}}(\bar{A}) \oplus \bar{R} \cong \bar{N}$. If we define $\bar{M} = \psi_{\#}\Omega_{\bar{R}}(\bar{A})$, then $\bar{M} \in \mathrm{Pic}(\bar{R})$ and $\bar{A}^{[n]} = S_{\bar{R}}(\bar{N}) \cong_{\bar{R}} S_{\bar{R}}(\bar{M})^{[n]}$. Let us set $M = R \otimes_{\bar{R}} \bar{M}$. Then $M \in \mathrm{Pic}(R)$ and $R \otimes_{\bar{R}} \bar{A}^{[n]} \cong_R S_R(M)^{[n]}$. Recall that $R \otimes_{\bar{R}} \bar{A} \cong A$, and hence $A^{[n]} \cong_R S_R(M)^{[n]}$. This shows that A is stably equivalent to $S_R(M)$.

(ii) $\Rightarrow$ (i): Obvious. □

6.8. Let R be a reduced ring, and let S be a reduced ring containing R. We say that R is F-closed in S if any element $a \in S$ such that $a^2, a^3, na \in R$ for some positive integer n is contained in R. When R is F-closed in any reduced ring S containing R, we say that R is an F-ring. An R-algebra A stably equivalent to $R^{[1]}$ is always isomorphic to $R^{[1]}$ if and only if R is an F-ring. Let C be a reduced ring with a finite number of minimal prime ideals, and let K be the total quotient ring of C. The intersection $F(C) = \bigcap_v C_v$ of all F-rings such that $C \subset C_v \subset K$ is also an F-ring, which is called an F-closure of C. We note that C is an F-ring if and only if $F(C) = C$. Suppose C is a subring of an F-ring R. Then there is an injective C-homomorphism $F(C) \hookrightarrow R$, and hence $F(C)$ may be viewed as a subring of R. (See [1].)

6.9. Corollary. *Let R be a ring such that $R/\mathfrak{r}$ is an F-ring, where $\mathfrak{r}$ is the nil-radical of R. Let A be an R-algebra. Then the following two conditions are equivalent:*

(i) *A is an invertible R-algebra such that $\Omega_R(A) \in \mathrm{Pic}(A)$.*

(ii) *$A \cong_R S_R(M)$ for some $M \in \mathrm{Pic}(R)$ (cf. [14]).*

Proof. (i) $\Rightarrow$ (ii): Let $\bar{A}$ be as in the proof of Theorem 6.7. Thus $\bar{A}^{[n]} \cong S_{\bar{R}}(\bar{M})^{[n]}$ for $\bar{M} \in \mathrm{Pic}(\bar{R})$. We set $C = \bar{R}/(\mathfrak{r} \cap \bar{R})$. Since $\bar{R}$ is a noetherian ring, C is a noetherian subring of the F-ring $R/\mathfrak{r}$. Therefore the F-closure $F(C)$ is also a subring of $R/\mathfrak{r}$ (6.8). If P is any prime ideal of $F(C)$, then $F(C)_P \otimes_{\bar{R}} \bar{A}$ is clearly stably equivalent to $F(C)_P^{[1]}$, and hence we have $F(C)_P \otimes_{\bar{R}} \bar{A} \cong F(C)_P^{[1]}$ because $F(C)_P$ is an F-ring, [1]. Thus we can take $N \in \mathrm{Pic}(F(C))$ so that $F(C) \otimes_{\bar{R}} \bar{A} \cong S_{F(C)}(N)$, [4]. Thus we get

$$R/\mathfrak{r} \otimes_R A \cong R/\mathfrak{r} \otimes_{F(C)} (F(C) \otimes_{\bar{R}} \bar{A}) \cong S_{R/\mathfrak{r}}(R/\mathfrak{r} \otimes_{F(C)} N).$$

Now let $\mathfrak{p}$ be any prime ideal of R. Then

$$(R_\mathfrak{p}/\mathfrak{r}R_\mathfrak{p}) \otimes_{R/\mathfrak{r}} (R/\mathfrak{r} \otimes_R A) \cong (R_\mathfrak{p}/\mathfrak{r}R_\mathfrak{p})^{[1]},$$

which implies $R_\mathfrak{p} \otimes_R A \cong R_\mathfrak{p}^{[1]}$, [1], because $R_\mathfrak{p} \otimes_R A$ is stably equivalent to $R_\mathfrak{p}^{[1]}$ (Theorem 6.7). This shows that $A \cong S_R(M)$ for some $M \in \mathrm{Pic}(R)$, [4], which completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Obvious. □

6.10. Corollary. *Let R be a ring containing an infinite field k. Let A and B be any pair of invertible R-algebras such that $\Omega_R(A) \in \mathrm{Pic}(A)$ and $\Omega_R(B) \in \mathrm{Pic}(B)$. Then $A \otimes_R B \cong S_R(M) \otimes_R S_R(N)$ for some $M, N \in \mathrm{Pic}(R)$.*

Proof. By virtue of Theorem 6.7, $A_\mathfrak{p}$ is stably equivalent to $R_\mathfrak{p}^{[1]}$ for each prime ideal $\mathfrak{p}$ of R. Similarly, $B_\mathfrak{p}$ is also stably equivalent to $R_\mathfrak{p}^{[1]}$. Thus $A_\mathfrak{p} \otimes_{R_\mathfrak{p}} B_\mathfrak{p} \cong R_\mathfrak{p}^{[2]}$, [2], i.e. $A \otimes_R B$ is a locally polynomial R-algebra. So we can choose a projective R-module L so that $A \otimes_R B \cong S_R(L)$, [4]. We identify $A \otimes_R B$ with $S_R(L)$. Then

$$\Omega_R(A \otimes_R B) \cong \Omega_R(A) \otimes_R B \oplus A \otimes_R \Omega_R(B) \cong S_R(L) \otimes_R L.$$

If $\phi : S_R(L) \to R$ be a natural augmentation, then

$$\phi_\#(\Omega_R(A) \otimes_R B) \oplus \phi_\#(A \otimes_R \Omega_R(B)) \cong L.$$

Let us set $M = \phi_\#(\Omega_R(A) \otimes_R B)$ and $N = \phi_\#(A \otimes_R \Omega_R(B))$. Then $M, N \in \mathrm{Pic}(R)$ and $A \otimes_R B = S_R(L) \cong S_R(M) \otimes_R S_R(N)$ as required. □

6.11. Remark. (See [22].) Let k be a field of characteristic zero, and let $k[t^2, t^3]$ be a subring of a polynomial ring $k[t]$. Let $P = (t^2, t^3)$ be a prime ideal of $k[t^2, t^3]$. We set $R = k[t^2, t^3]_P$ and $A = R[X + tX^n] + PR[X]$ $(n > 1)$, where X is an indeterminate. Then A is weakly projective R-algebra such that $A \ncong R[X]$, [22, p. 355]. Note that R is an F-ring, and hence A is not invertible (Corollary 6.9).

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